LATTICE ARTIFACTS PROPORTIONAL TO THE QUARK MASS IN THE QCD RUNNING COUPLING

M. Costa [1,2,3], D. Gavriel [2,4,*], H. Panagopoulos [2], G. Spanoudes [2]

[1] Cyprus University of Technology, [2] University of Cyprus, [3] Rinnoco Ltd, [4] Università di Parma and INFN [*] Speaker

Abstract

Discretization artifacts proportional to the quark mass can limit the accuracy of strong-coupling determinations in lattice QCD, especially when heavy quarks are present. In this work, we perform a lattice perturbative analysis of such $\mathcal{O}(am)$ effects in the running coupling by computing the two-loop renormalization factor Z_g. Using the background field method together with clover-improved Wilson fermions and Symanzik-improved gauge actions, we determine the mass-dependent components of the relevant two-point Green's functions and obtain the improvement coefficients needed to remove $\mathcal{O}(am)$ artifacts in mass-independent renormalization schemes. Our results are presented for general values of the number of colors N_c, the number of quark flavors N_f, and the clover coefficient $c_{\rm sw}$, and satisfy all symmetry and consistency constraints. Numerical values are provided for widely used gauge actions, enabling improved control of mass-related cutoff effects in high-precision determinations of the strong coupling constant from lattice QCD.
For the full paper [1], please scan the QR code in the Acknowledgments Section.

Motivation

- High-precision determinations of the strong coupling α_s rely significantly on lattice QCD [2, 3].
- In studies involving heavy quarks or decoupling strategies for scale evolution, values of am_q become large → mass-dependent discretization errors of $\mathcal{O}(am)$ can no longer be neglected.
- Mass-independent renormalization schemes require the introduction of a modified bare coupling $\tilde{g}_0^2$ [4], whose definition depends on the improvement coefficient $b_g(g_0^2)$.
- Currently, $b_g(g_0^2)$ is only known to one loop in perturbation theory → introduces a non-negligible uncertainty in precision determinations of α_s [5]; hence, a two-loop determination is essential.

Computational Setup and Methods

We compute the two-loop renormalization factor Z_g, which relates the bare and renormalized couplings:

$$g_0 = Z_g(g_0^2, a\bar{\mu})\, g,$$

using the background field method [6, 7] in lattice perturbation theory.

- Lattice actions:
  - Fermion action: Clover improved action
  - Gauge action: Symanzik-improved family [Wilson, Tree-Level Symanzik(TLS), Iwasaki]
- Background field decomposition:

$$U_\mu(x) = U_\mu^Q(x)\, U_\mu^B(x), \qquad U_\mu^Q(x) = e^{ig_0 Q_\mu(x)}, \quad U_\mu^B(x) = e^{iaB_\mu(x)}.$$

  Here, Q_μ denotes the quantum gauge field and B_μ the external background field.
- Background field renormalization [8]:

$$Z_B(g_0^2, a\mu)\, Z_g^2(g_0^2, a\mu) = 1.$$

  In this framework, it suffices to calculate Z_B from the 1-particle-irreducible (1-PI) 2-point Green's function of background field, both in the continuum and on the lattice, in the presence of a fermion mass.
- To ensure $\mathcal{O}(a)$ improvement, we define a modified bare coupling:

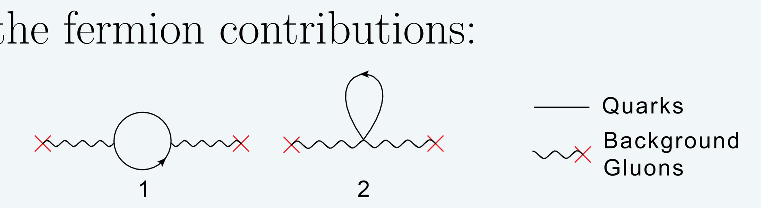
$$\tilde{g}_0^2 = g_0^2\Big(1 + b_g(g_0^2)\, am_q\Big),$$

  with subtracted quark mass $m_q = m_0 - m_c(g_0^2)$, where m_c is the critical mass.

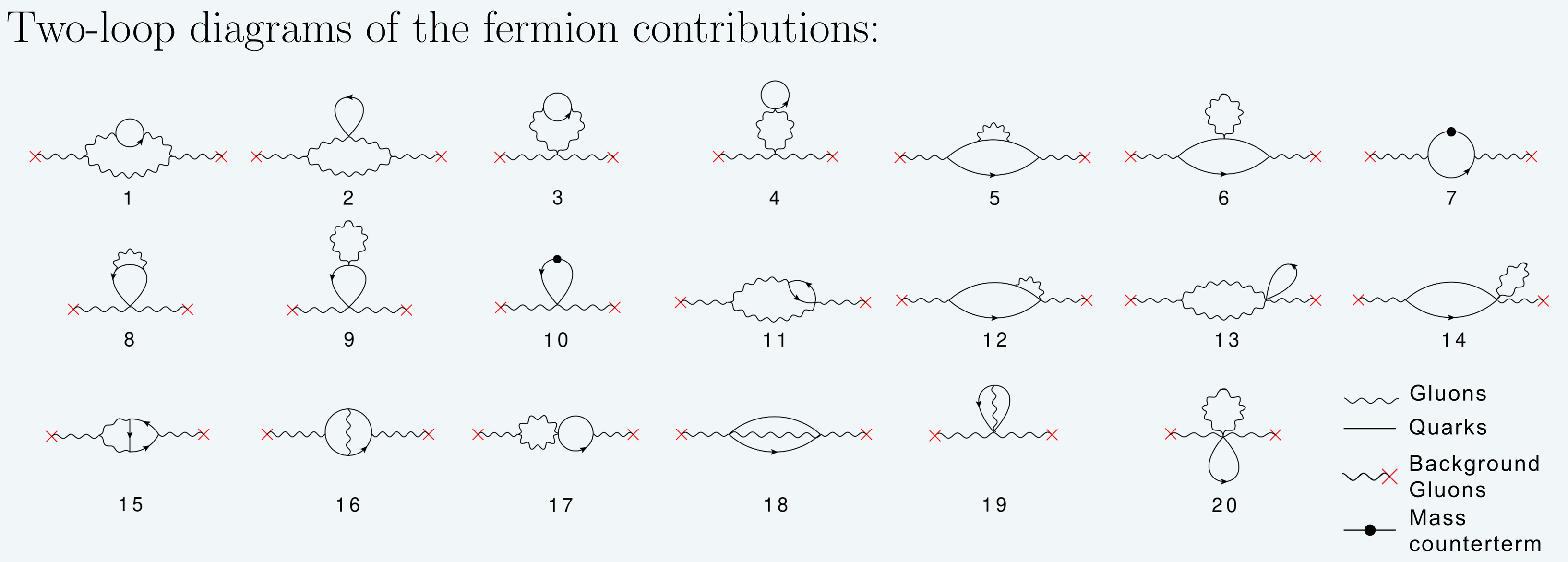
Feynman Diagrams

The mass effects that contribute to the 2-point Green's functions are associated with Feynman diagrams containing at least one fermion line.

- One-loop diagrams of the fermion contributions:



- Two-loop diagrams of the fermion contributions:



Each diagram is meant to be symmetrized over the color indices, Lorentz indices, and momenta of the two external background fields.

Results for b_g up to Two-Loop Order

Expanding $b_g(g_0)$ in the modified bare coupling gives:

$$\tilde{g}_0^2 = g_0^2\left[1 + am_q\Big(b_g^{(1)} g_0^2 + b_g^{(2)} g_0^4 + \mathcal{O}(g_0^6)\Big)\right].$$

- The first coefficient is:

$$b_g^{(1)} = N_f\left[0.0272837(1) - 0.0223503(1)\, c_{\rm sw} + 0.0070667(1)\, c_{\rm sw}^2 - (1 - c_{\rm sw})\frac{2}{16\pi^2}\ln(a^2p^2)\right],$$

  independent of the gluon action and the number of colors.
- The second coefficient for the TLS action is:

$$\begin{aligned}
b_g^{(2)}|_{\rm TLS} = \frac{N_f}{N_c}\Bigg\{&(1 - c_{\rm sw})\left(\frac{\ln(a^2p^2)}{16\pi^2}\right)^2 - 0.0051236(24) - 0.013840(4)c_{\rm sw} \\
&+ 0.0083484(19)c_{\rm sw}^2 + 0.00029588(7)c_{\rm sw}^3 - 0.000001975(4)c_{\rm sw}^4 \\
&+ \Big(0.0576153(9) + 0.0757940(14)c_{\rm sw} - 0.0367695(17)c_{\rm sw}^2 + 0.01954606(7)c_{\rm sw}^3\Big)\frac{\ln(a^2p^2)}{16\pi^2}\Bigg\} \\
+ N_f N_c\Bigg\{&(1 - c_{\rm sw})\left(\frac{\ln(a^2p^2)}{16\pi^2}\right)^2 - 0.007598(6) + 0.003048(6)c_{\rm sw} \\
&- 0.002202(5)c_{\rm sw}^2 - 0.00002829(13)c_{\rm sw}^3 + 0.0000156943(35)c_{\rm sw}^4 \\
&+ \Big(- 0.130706(33) - 0.056392(19)c_{\rm sw} + 0.059879(5)c_{\rm sw}^2 - 0.0165255904(23)c_{\rm sw}^3\Big)\frac{\ln(a^2p^2)}{16\pi^2}\Bigg\}.
\end{aligned}$$

- In the above Equation only the tree-level value of $c_{\rm sw}$ contributes; higher-order effects enter through $b_g^{(1)}$ once the perturbative expansion $c_{\rm sw} = 1 + g_0^2 c_{\rm sw}^{(1)}$ is used. For $N_c = 3$, and using the values of $c_{\rm sw}^{(1)}$ [9, 10] corresponding to the Symanzik-improved gluon actions used in this work, we obtain:

$$b_g = \begin{cases} N_f\Big(0.012000(1)\, g_0^2 - 0.020067(20)\, g_0^4 + \mathcal{O}(g_0^6)\Big), & \text{Wilson,} \\ N_f\Big(0.012000(1)\, g_0^2 - 0.025347(30)\, g_0^4 + \mathcal{O}(g_0^6)\Big), & \text{TLS,} \\ N_f\Big(0.012000(1)\, g_0^2 - 0.04201(6)\, g_0^4 + \mathcal{O}(g_0^6)\Big), & \text{Iwasaki.} \end{cases}$$

  These expressions are free of logarithmic terms and thus independent of the external momentum p. Comparison with nonperturbative results [11] shows, however, that sizable differences persist even at two loops.

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